«تمرینات فصل اول»

بسمه تعالی

۱- کدام یک از توابع داده شده در بازه‌های تعریف شده پیوسته تکه‌ای هستند.

a) $f(x)=\begin{cases} x & ; \quad 0<x<1 \\ 0 & ; \quad 1<x<2 \end{cases}$

پیوسته تکه‌ای هست

b) $f(x)=\begin{cases} 0 & ; \quad -1<x\leq 0 \\ \frac{1}{x} & ; \quad 0<x<1 \end{cases}$

چون در $x=0$ ناپیوستگی از نوع بی‌نهایت دارد پیوسته تکه‌ای نیست

c) $f(x)=\begin{cases} 2 & ; \quad 0\leq x<1 \\ x^2 & ; \quad 1<x<2 \end{cases}$

پیوسته تکه‌ای هست

d) $f(x)=\begin{cases} 1-x & ; \quad -1\leq x<2 \\ \frac{x}{2-x} & ; \quad 2<x\leq 3 \end{cases}$

چون در $x=2$ ناپیوستگی از نوع بی‌نهایت دارد پیوسته تکه‌ای نیست

۲- کدام یک از توابع زیر زوج، فرد، یا نه زوج و نه فرد هستند. فرد $f(-x)=-f(x)$ زوج $f(x)=f(-x)$

a) $x+2x^2+3x^3 \qquad f(-x)=-x+2(-x)^2+3(-x)^3=-x+2x^2-3x^3$ نه زوج نه فرد

$f(x)=x\ln x \qquad f(-x)=(-x)\ln(-x)$ دامنه لگاریتم $(0,\infty)$ است یعنی تابع نه زوج است نه فرد

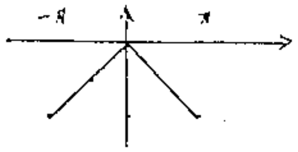
$f(x)=\frac{1}{x} \qquad f(-x)=-\frac{1}{x}=-f(x)$ فرد است

$f(x)=\sinh x \qquad f(-x)=\frac{e^{-x}-e^{x}}{2}=-\frac{e^{x}-e^{-x}}{2}=-\sinh x$ فرد است

$f(x)=e^{x} \qquad f(-x)=e^{-x}$ نه زوج نه فرد

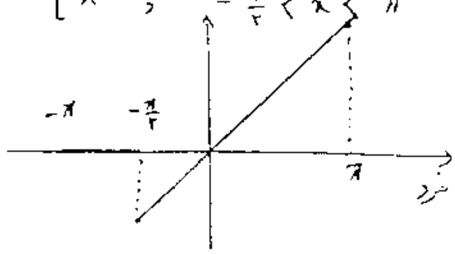
$f(x)=e^{|x|} \qquad f(-x)=e^{|-x|}=e^{|x|}$ زوج

b) $f(x)=\begin{cases} x & ; \quad -\pi<x<0 \\ -x & ; \quad 0<x<\pi \end{cases}$



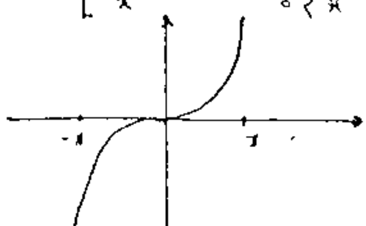
تابع زوج

c) $f(x)=\begin{cases} 0 & ; \quad -\pi<x<-\frac{\pi}{2} \\ x & ; \quad -\frac{\pi}{2}<x<\pi \end{cases}$



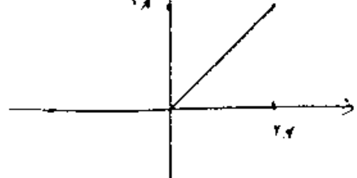
نه زوج نه فرد

d) $f(x)=\begin{cases} -x^2 & \quad -\pi<x<0 \\ x^2 & \quad 0<x<\pi \end{cases}$



تابع فرد

e) $f(x)=|x| \quad 0<x<2\pi$ نه زوج نه فرد



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الف) $\int_a^{a+p} f(x)\,dx = \int_b^{b+p} f(x)\,dx$

می‌دانیم $\int_a^{a+2p} f(x)\,dx = \int_{-p}^{p} f(x)\,dx$ ① $\int_b^{b+2p} f(x)\,dx = \int_{-p}^{p} f(x)\,dx$ ②

$$①,② \Rightarrow \int_a^{a+2p} f(x)\,dx = \int_b^{b+2p} f(x)\,dx \Rightarrow \int_a^{a+p} f(x)\,dx + \int_{a+p}^{a+2p} f(x)\,dx = \int_b^{b+p} f(x)\,dx + \int_{b+p}^{b+2p} f(x)\,dx \quad (*)$$

$$\int_{a+p}^{a+2p} f(x)\,dx = \int_{(a+p)-p}^{(a+2p)-p} f(x-p)\,dx = \int_a^{a+p} f(x)\,dx$$ (به علت متناوب بودن با دوره p)

$$\Rightarrow \int_{a+p}^{a+2p} = \int_a^{a+p} f(x)\,dx$$ به همین ترتیب $$\Rightarrow \int_{b+p}^{2p+b} f(x)\,dx = \int_b^{b+p} f(x)\,dx$$

$$(*) \Rightarrow 2\int_a^{a+p} f(x)\,dx = 2\int_b^{b+p} f(x)\,dx \Rightarrow \int_a^{a+p} f(x)\,dx = \int_b^{b+p} f(x)\,dx$$

ب) $$\int_a^{a+2p} f(x)\,dx = \int_a^{0} f(x)\,dx + \int_0^{p} f(x)\,dx + \int_p^{2p} f(x)\,dx + \int_{2p}^{2p+a} f(x)\,dx = \cancel{-\int_0^{a} f(x)\,dx} +$$

$$\int_0^{p} f(x)\,dx + \int_{-p}^{0} \underbrace{f(x-2p)}_{f(x)}\,dx + \int_0^{a} \underbrace{f(x-2p)}_{f(x)}\,dx = \int_{-p}^{p} f(x)\,dx$$

۴- سری فوریه هریک از توابع زیر را بیابید.

a) $f(x) = x + \sin x \qquad -\pi < x < \pi$

$f(x) = g(x) + \sin x$

$g(x) = x \qquad -\pi < x < \pi \qquad a_n = 0$ به علت فرد بودن g

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx \Rightarrow \pi b_n = 2\int_0^{\pi} x \sin nx\,dx = 2\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$

$$b_n = \frac{2\pi}{n}(-1)^{n+1} \qquad g(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx \Rightarrow f(x) = \sin x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

b) $\begin{cases} f(x) = \sin\frac{\pi x}{l} & ; \quad 0 < x < l \\ f(x) = f(-x) & ; \quad -l < x < 0 \end{cases}$ $\qquad b_n = 0 \qquad$ به علت زوج بودن f

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{4}{\pi} \quad , a_1 = 0$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi}{l}x\,dx \qquad n \geq 2$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \Rightarrow 2\sin\frac{\pi}{l}x\cos\frac{n\pi}{l}x = \sin(1+n)\frac{\pi}{l}x + \sin(1-n)$$

$$l\,a_n = \int_0^l \left(\sin(1+n)\frac{\pi}{l}x + \sin(1-n)\frac{\pi}{l}x\right)dx = \frac{-l}{\pi(1+n)}\cos(1+n)\frac{\pi}{l}x\Big]_0^l - \frac{l}{\pi(1-n)}\cos(1-$$

$$a_n = \frac{-1}{\pi(1+n)}\left[(-1)^{n+1} - 1\right] - \frac{1}{\pi(1-n)}\left[(-1)^{1-n} - 1\right] = -\frac{1}{\pi}\left[(-1)^{n+1} - 1\right]\left(\frac{1}{1+n} + \frac{1}{1-}\right.$$

$$a_n = \frac{2\left[(-1)^n + 1\right]}{\pi(1-n^2)} \Rightarrow f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{1-n^2}\cos\frac{n\pi}{l}x$$

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c) $f(x) = \sinh x \quad ; \quad -1 < x < 1$ $\qquad$ f فرد است پس $a_n = 0$

$$b_n = \frac{1}{1}\int_{-1}^{1}\sinh x\sin n\pi x\,dx = 2\int_0^1 \frac{e^x - e^{-x}}{2}\sin n\pi x\,dx = \int_0^1 e^x\sin n\pi x\,dx - \int_0^1 e^{-x}\sin n\pi x\,dx = I$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$ * به روش جزء به جزء می‌آییم :

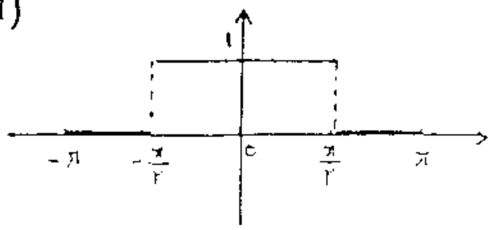
$$I_1 = \frac{e^x}{1+(n\pi)^2}\left(\sin n\pi x - n\pi\cos n\pi x\right)\Big]_0^1 = \frac{n\pi}{1+(n\pi)^2}\left(e^1(-1)^{n+1} + 1\right).$$

$$I_2 = \frac{e^{-x}}{1+(n\pi)^2}\left(-\sin n\pi x - n\pi\cos n\pi x\right)\Big]_0^1 = \frac{n\pi}{1+(n\pi)^2}\left(e^{-1}(-1)^{n+1} + 1\right).$$

$$I_1 - I_2 = \frac{n\pi}{1+(n\pi)^2}\left((e^1 - e^{-1})(-1)^{n+1}\right) = \frac{2n\pi\sinh 1}{1+(n\pi)^2}(-1)^{n+1} \qquad \sinh 1 = \frac{e^1 - e^{-1}}{2}$$

$$\Rightarrow f(x) = 2\pi\sinh 1\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{1+(n\pi)^2}\sin n\pi x$$

d)



$$f(x)=\begin{cases}0 & ; \quad -\pi \leqslant x \leqslant -\frac{\pi}{2}\\ 1 & ; \quad -\frac{\pi}{2} \leqslant x < \frac{\pi}{2}\\ 0 & ; \quad \frac{\pi}{2} \leqslant x \leqslant \pi\end{cases}$$

$f(x)$ زوج است

پس $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \frac{\pi}{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx = \frac{1}{n\pi}\sin nx\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx = \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right)$$

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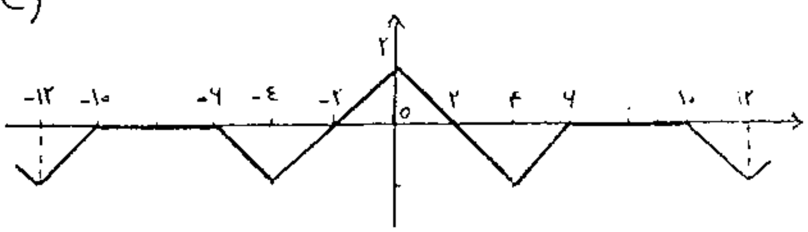
e)



$$f(x)=\begin{cases}2-x & ; \quad 0 \leqslant x < 4\\ -6+x & \quad\ 4 \leqslant x < 6\\ 0 & \quad\ 6 \leqslant x < 8\end{cases}$$

$b_n = 0$: زوج است پس $f(x)$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx = \frac{2}{8}\int_0^8 f(x)\,dx = \frac{1}{4}\int_0^4 f(x)\,dx + \frac{1}{4}\int_4^6 f(x)\,dx + \frac{1}{4}\int_6^8 f(x)\,dx$$

$$= 0 + \frac{1}{4}\left(\frac{-2\times 2}{2}\right) + 0 = -\frac{1}{2}$$

$$a_n = \frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi}{8}x\,dx = \frac{1}{4}\int_0^4 (2-x)\cos\frac{n\pi}{8}x\,dx + \frac{1}{4}\int_4^6 (x-6)\cos\frac{n\pi}{8}x\,dx + 0$$

$$= \frac{1}{4}\left[(2-x)\frac{8}{n\pi}\sin\frac{n\pi}{8}x - \left(\frac{8}{n\pi}\right)^2\cos\frac{n\pi}{8}x\right]_0^4 +$$

$$\frac{1}{4}\left[(x-6)\frac{8}{n\pi}\sin\frac{n\pi}{8}x + \left(\frac{8}{n\pi}\right)^2\cos\frac{n\pi}{8}x\right]_4^6 = \frac{16}{(n\pi)^2}\left[1 - 2\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{4}\right]$$

$$\Rightarrow f(x) = -\frac{1}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - 2\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{4}\right)\cos\frac{n\pi}{8}x$$

f) $f(x) = \begin{cases} \frac{1}{2} & ; \quad -1 < x < 0 \\ -x & ; \quad 0 < x < 1 \end{cases}$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}\frac{1}{2}\,dx + \int_{0}^{1} -x\,dx = \frac{1}{2} - \frac{1}{2} = 0$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\frac{1}{2}\cos n\pi x\,dx + \int_{0}^{1} -x\cos n\pi x\,dx = -\Big[\frac{x}{n\pi}\sin n\pi x + \frac{1}{(n\pi)^2}\cos$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\frac{1}{2}\sin n\pi x\,dx + \int_{0}^{1} -x\sin n\pi x\,dx$$

$$a_n = \frac{1-(-1)^n}{(n\pi)^2} \qquad\qquad b_n = -\frac{1+(-1)^n}{2\pi} + \frac{(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \frac{(-1)^n+1}{2}\sin n\pi x$$

- با بکار بردن سری فوریه $f(x) = \begin{cases} 1 & ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & ; \quad \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$ نشان دهید $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

$$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\times dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 0\,dx = 1 \qquad\qquad b_n = 0$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\cos nx\,dx = \frac{2}{n\pi}\sin nx\Big]_{0}^{\frac{\pi}{2}} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx = \frac{f(x^+)+f(x^-)}{2}$$ طبق قضیه دیریکله :

$$x = \pi \Rightarrow \frac{1}{2} + \frac{2}{\pi}\Big(\cos\pi - \frac{1}{3}\cos 3\pi + \frac{1}{5}\cos 5\pi - \cdots\Big) = \frac{f(\pi^+)+f(\pi^-)}{2} = 0$$

$$\frac{1}{2} + \frac{2}{\pi}\Big(-1 + \frac{1}{3} - \frac{1}{5} + \cdots\Big) = 0 \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$$

۴) سری فوریه کسینوسی متناظر با هر یک از توابع زیر را بیابید.

a) $$f(x)=\begin{cases} 0 & ; \quad 0<x<\frac{l}{2} \\ 1 & ; \quad \frac{l}{2}<x<l \end{cases}$$

$$a_0=\frac{2}{l}\int_{\frac{l}{2}}^{l} dx=\frac{2}{l}\times\frac{l}{2}=1 \Rightarrow a_0=1$$

$$a_n=\frac{2}{l}\int_{\frac{l}{2}}^{l}\cos\frac{n\pi}{l}x\,dx=\frac{2}{l}\left[\frac{l}{n\pi}\sin\frac{n\pi}{l}x\right]_{\frac{l}{2}}^{l}=\frac{2}{\pi}\left(-\frac{1}{n}\right)\sin\frac{n\pi}{2}$$

$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}-\frac{2}{\pi}\times\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi}{l}x$$

$$=\frac{1}{2}-\frac{2}{\pi}\left(\cos\frac{\pi x}{l}-\frac{1}{3}\cos\frac{3\pi}{l}x+\cdots\right)$$

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b) $$f(x)=\sin\frac{\pi x}{l} \qquad 0<x<l$$

$$a_0=\frac{2}{l}\int_0^l\sin\frac{\pi x}{l}\,dx=\frac{2}{\pi}\left[-\cos\frac{\pi x}{l}\right]_0^l=\frac{4}{\pi}$$

$$a_1=\frac{2}{l}\int_0^l\sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\,dx=\frac{1}{l}\left(\frac{l}{2\pi}\cos\frac{2\pi x}{l}\right)_0^l=0$$

$$a_n=\frac{2}{l}\int_0^l\sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l\frac{1}{2}\left[\sin(1+n)\frac{\pi}{l}x+\sin(1-n)\frac{\pi}{l}x\right]dx$$

$$=-\frac{1}{l}\left[\frac{l}{\pi(1+n)}\cos(1+n)\frac{\pi x}{l}+\frac{l}{\pi(1-n)}\cos(1-n)\frac{\pi x}{l}\right]_0^l$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{1+n}+\frac{(-1)^n}{1-n}+\frac{1}{1+n}+\frac{1}{1-n}\right]=\frac{-2\left[(-1)^n+1\right]}{\pi(n^2-1)}$$

$$f(x)=\frac{2}{\pi}-\frac{2}{\pi}\left(\frac{1}{1\times3}\cos\frac{2\pi x}{l}+\frac{1}{3\times5}\cos\frac{4\pi x}{l}+\cdots\right)$$

c) $f(x) = \sin x \qquad 0 < x < \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{4}{\pi} \qquad\qquad a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right] dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{1+n} + \frac{(-1)^{n+1}}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\,\frac{2}{1-n^2}\left[(-1)^n + 1\right]$$

$$f(x) = \frac{4}{\pi}\left(\frac{1}{2} + \frac{\cos 2x}{1-2^2} + \frac{\cos 4x}{1-4^2} + \dots\right)$$

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d) $f(x) = \begin{cases} x & ; \quad 0 < x \leq 1 \\ 2-x & ; \quad 1 < x < 2 \end{cases}$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\, dx = \int_0^1 x\, dx + \int_1^2 (2-x)\, dx = 1$$

$$a_n = \frac{2}{2}\int_0^1 x \cos\frac{n\pi}{2}x \, dx + \frac{2}{2}\int_1^2 (2-x)\cos\frac{n\pi}{2}x\, dx$$

$$= \left[\frac{2}{n\pi} x \sin\frac{n\pi}{2}x + \left(\frac{2}{n\pi}\right)^2 \cos\frac{n\pi x}{2}\right]_0^1 +$$

$$\left[2 \times \frac{2}{n\pi}\sin\frac{n\pi}{2}x - \frac{2}{n\pi}x\sin\frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2}x\right]_1^2$$

$$= \Big[\cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2} - \left(\frac{2}{n\pi}\right)^2 + \cancel{\frac{4}{n\pi}\sin n\pi} - \cancel{\frac{4}{n\pi}\sin n\pi} - \left(\frac{2}{n\pi}\right)^2\cos n\pi$$

$$- \cancel{\frac{4}{n\pi}\sin\frac{n\pi}{2}} + \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2}\Big] = \left(\frac{2}{n\pi}\right)^2\left(2\cos\frac{n\pi}{2} - \cos n\pi \dots\right.$$

اگر n فرد باشد $a_n=0$ است. همچنین اگر n مضرب ۴ باشد داریم $a_n=0$ در غیر این صورت $a_n = -\frac{16}{(n\pi)^2}$

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)\cos\frac{n\pi}{2}x$$

n می‌تواند مقادیر ... ۶، ۱۰، ۱۴، ۱۸ و ... را اختیار کند

$$f(x) = \frac{1}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x \quad \Leftarrow$$

۷) سری فوریه سینوسی متناظر با هر یک از توابع زیر را به دست آورید؟

a) $f(x) = \begin{cases} x & ; \quad 0 < x < \frac{L}{2} \\ L - x & ; \quad \frac{L}{2} < x < L \end{cases}$

برای تمام توابع این سؤال $a_n = 0$ است.

$$b_n = \frac{2}{L}\int_0^{L/2} x \sin\frac{n\pi}{L}x\, dx + \frac{2}{L}\int_{L/2}^{L} (L-x)\sin\frac{n\pi}{L}x\, dx$$

$$= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi}{L}x + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{L}x\right]_0^{\frac{L}{2}} -$$

$$\frac{2}{L}\left[(L-x)\frac{L}{n\pi}\cos\frac{n\pi}{L}x + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{L}x\right]_{L/2}^{L}$$

$$\Rightarrow f(x) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)\frac{\pi}{L}x$$

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b) $f(x) = \begin{cases} x & ; \quad 0 < x < 1 \\ 2 - x & ; \quad 1 < x < 2 \end{cases}$

حالت خاص a با $L=2$ می باشد پس داریم:

$$f(x) = \frac{4 \times 2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)\frac{\pi x}{2}$$

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c) $f(x) = \cos 2x \qquad 0 < x < \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\cos 2x \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin(n+2)x + \sin(n-2)x\right)dx$$

$$= \left[-\frac{1}{\pi(n+2)}\cos(n+2)x\right]_0^{\pi} + \left[-\frac{1}{\pi(n-2)}\cos(n-2)x\right]_0^{\pi} = \frac{(-1)^{n+2}-1}{\pi(n+2)} - \frac{(-1)^{n-2}-1}{\pi(n-2)}$$

$$= \frac{2n\left[1+(-1)^{n+1}\right]}{\pi(n^2-4)}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi}\times\frac{1+(-1)^{n+1}}{n^2-4}\times n\sin nx = \frac{4}{\pi}\left[\frac{\sin x}{1^2-2^2} + \frac{3\sin 3x}{3^2-2^2} + \cdots\right]$$

d) $f(x) = x^2 \quad ; \quad 0 < x < \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx = \frac{2}{\pi}\left(-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right)_0^{\pi}$$

$$= \left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right)\frac{2}{\pi} = \left(\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\left((-1)^n - 1\right)\right)\frac{2}{\pi}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{\pi^2}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]\right]\sin nx$$

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۸ - هرگاه $-\pi < x < \pi \; ; \; f(x) = \cos\mu x$ که در آن μ عددی غیرصحیح است. آنگاه نشان دهید:

(الف $f(x) = \frac{2\mu}{\pi}\sin\mu\pi\left\{\frac{1}{2\mu^2} + \frac{\cos x}{1-\mu^2} - \frac{\cos 2x}{2^2-\mu^2} + \cdots + \frac{(-1)^{n+1}\cos nx}{n^2-\mu^2} + \cdots\right\}$

ب) از آن استنتاج کنید $\cot\mu\pi = \frac{2\mu}{\pi}\left\{\frac{1}{2\mu^2} + \frac{1}{\mu^2-1} + \frac{1}{\mu^2-2^2} + \cdots + \frac{1}{\mu^2-n^2} + \cdots\right\}$

ج) همچنین نشان دهید که $\sum_{n=1}^{\infty}\frac{1}{9n^2-1} = \frac{1}{2} - \frac{\pi\sqrt{3}}{18}$

الف) تابع زوج است پس داریم: $b_n = 0$ و $a_0 = \frac{2}{\pi}\int_0^{\pi}\cos\mu x \, dx = \frac{2}{\mu\pi}\sin\mu\pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos\mu x\cos nx \, dx = \frac{2}{\pi}\times\frac{1}{2}\int_0^{\pi}\left[\cos(\mu+n)x + \cos(\mu-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{1}{\mu+n}\sin(\mu+n)x + \frac{1}{\mu-n}\sin(\mu-n)x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{\mu+n}\sin(\mu\pi+n\pi) + \frac{1}{\mu-n}\sin(\mu\pi-n\pi)\right]$$

$$= \frac{\sin\mu\pi\cos n\pi}{\pi}\left[\frac{1}{\mu+n} + \frac{1}{\mu-n}\right] = \frac{2\mu\sin\mu\pi}{\pi(n^2-\mu^2)}(-1)^{n+1}$$

$$\Rightarrow f(x) = \frac{2\mu\sin\mu\pi}{\pi}\left[\frac{1}{2\mu^2} + \frac{\cos x}{1-\mu^2} - \frac{\cos 2x}{2^2-\mu^2} + \cdots + \frac{(-1)^{n+1}\cos nx}{n^2-\mu^2} + \cdots\right]$$

ب) اگر $x=\pi$ داریم:

$$\cos\mu\pi = \frac{2\mu}{\pi}\sin\mu\pi\left[\frac{1}{2\mu^2}+\frac{\cos\pi}{1-\mu^2}-\frac{\cos 2\pi}{2^2-\mu^2}+\cdots\right]$$

$$= \frac{2\mu}{\pi}\sin\mu\pi\left[\frac{1}{2\mu^2}+\frac{-1}{1-\mu^2}+\frac{-1}{2^2-\mu^2}+\cdots+\frac{-1}{n^2-\mu^2}+\cdots\right]$$

$$\cot\mu\pi = \frac{\cos\mu\pi}{\sin\mu\pi} = \frac{2\mu}{\pi}\left[\frac{1}{2\mu^2}+\frac{1}{\mu^2-1}+\frac{1}{\mu^2-2^2}+\cdots+\frac{1}{\mu^2-n^2}+\cdots\right]$$

ج) $\mu=\frac{1}{3} \Leftarrow$

$$\cot\frac{\pi}{3} = \frac{2}{3\pi}\left[\frac{9}{2}+\frac{9}{1-9}+\frac{9}{1-9\times 2^2}+\cdots+\frac{9}{1-9n^2}+\cdots\right]$$

$$\frac{\sqrt{3}}{3} = \frac{18}{3\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1-9n^2}\right] \Rightarrow \frac{\sqrt{3}\,\pi}{18} = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1-9n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{9n^2-1} = \frac{1}{2}-\frac{\sqrt{3}\,\pi}{18}$$

۹- ثابت کنید به ازای $-\pi<x<\pi$ (الف

$$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\cdots\right)$$

که در آن a غیرصحیح است

ب)

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\sin nx$$

الف) $\sin ax$ فرد است پس $a_0=a_n=0$ و

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx$$

$$= \frac{1}{2}\times\frac{2}{\pi}\int_0^{\pi}\cos(a-n)x-\cos(a+n)x\,dx = \frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n}-\frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(a\pi-n\pi)}{a-n}-\frac{\sin(a\pi+n\pi)}{a+n}\right] = \frac{\sin a\pi\cos n\pi}{\pi}\left[\frac{1}{a-n}-\frac{1}{a+n}\right]$$

$$= \frac{2\sin a\pi}{\pi}\times\frac{n(-1)^n}{(a^2-n^2)}$$

$$\Rightarrow \sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{1}{1-a^2}\sin x-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\cdots\right]$$

ب) $x\cos x$ تابعی فرد است پس داریم: $a_n = a_0 = 0$ و

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[x\sin(n+1)x + x\sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \frac{1}{(n+1)^2}\sin(n+1)x - \frac{x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \left[\frac{\pi(-1)^n}{n+1} + 0 + \frac{\pi(-1)^n}{n-1} + 0 + 0 - 0 + 0 - 0\right]\frac{1}{\pi} = \frac{2n(-1)^n}{n^2-1} \qquad n \neq 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x = -\frac{1}{2}$$

$$f(x) = x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\sin nx$$

۱۰- ثابت کنید که (فرمول پارسوال)

$$\int_{-l}^{l}[f(x)]^2 dx = l\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)$$

در این فرمول a_n و b_n ضرایب اویلر سری فوریه تابع $f(x)$ هستند.

برهان: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos kx + b_n\sin kx$

$$k = \frac{n\pi}{l}$$

$$f^2(x) = \frac{a_0^2}{4} + a_0\sum_{n=1}^{\infty} a_n\cos kx + b_n\sin kx + \left[\sum_{n=1}^{\infty} a_n\cos kx + \sum_{n=1}^{\infty} b_n\sin kx\right]$$

$$\left[\sum_{n=1}^{\infty} a_n\cos kx + \sum_{n=1}^{\infty} b_n\sin kx\right]$$

$$\int_{-l}^{l} f^2(x)\, dx = A + B + C \qquad A = \int_{-l}^{l}\frac{a_0^2}{4}dx = \frac{a_0^2}{2}l$$

$$B = \int_{-l}^{l}\left(a_0\sum_{n=1}^{\infty} a_n\cos kx + b_n\sin kx\right) dx = \int_{-l}^{l}\left(a_0\sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right)$$

$$= a_0\sum_{n=1}^{\infty}\left(\int_{-l}^{l} a_n\cos\frac{n\pi}{l}x\, dx + \int_{-l}^{l} b_n\sin\frac{n\pi}{l}x\, dx\right) = 0 + 0 = 0$$

$$C = \int_{-l}^{l}\left[\sum_{n=1}^{\infty} a_n\cos kx + \sum_{n=1}^{\infty} b_n\sin kx\right]\left[\sum_{n=1}^{\infty} a_n\cos kx + \sum_{n=1}^{\infty} b_n\sin kx\right] dx$$

$$= \int_{-l}^{l} \left(\sum_{n=1}^{\infty} a_n \cos kx\right)^2 dx + 2\int_{-l}^{l} \sum_{n=1}^{\infty} a_n \cos kx \cdot \sum_{n=1}^{\infty} b_n \sin kx \, dx + \int_{-l}^{l} \left(\sum_{n=1}^{\infty} b_n \sin kx\right)^2 dx$$

$$= I_1 + I_2 + I_3 \qquad I_1 = \int_{-l}^{l} \left(a_1 \cos\frac{\pi}{l}x + a_2 \cos\frac{2\pi}{l}x + \cdots\right)\left(a_1 \cos\frac{\pi}{l}x + a_2 \cos\frac{2\pi}{l}x + \cdots\right) dx$$

عبارت زیرانتگرال I_1 از ضرب عوامل $a_m \cos\frac{m\pi}{l}x$ و $a_n \cos\frac{n\pi}{l}x$ حاصل می‌شود دو حالت در (او له) دارد.

(حالت اول) $m = n \Rightarrow \int_{-l}^{l} a_n \cos\frac{n\pi}{l}x \cdot a_n \cos\frac{n\pi}{l}x \, dx = a_n^2 \int_{-l}^{l} \cos^2\frac{n\pi}{l}x \, dx = a_n^2 \cdot l$

(حالت دوم) $m \neq n \Rightarrow \int_{-l}^{l} a_n \cos\frac{n\pi}{l}x \cdot a_m \cos\frac{m\pi}{l}x \, dx = a_n a_m \int_{-l}^{l} \cos\frac{n\pi}{l}x \cdot \cos\frac{m\pi}{l}x = 0$

با توجه به روابط صفحه‌ی ۲ کتاب شیدفر

$$\Rightarrow I_1 = l \sum_{n=1}^{\infty} a_n^2$$

$$I_2 = 2\int_{-l}^{l} \sum_{n=1}^{\infty} a_n \cos kx \cdot \sum_{n=1}^{\infty} b_n \sin kx = 2\int_{-l}^{l} \left(a_1 \cos\frac{\pi}{l}x + a_2 \cos\frac{2\pi}{l}x + \cdots\right)\left(b_1 \sin\frac{\pi}{l}x + \cdots\right) dx$$

$$= 2\int_{-l}^{l} \sum_{m,n=1}^{\infty} a_m \cos\frac{m\pi}{l}x \cdot b_n \sin\frac{n\pi}{l}x = 2\sum_{m,n=1}^{\infty} \int_{-l}^{l} a_m b_n \cos\frac{m\pi}{l}x \sin\frac{n\pi}{l}x \, dx = 2\times 0 = 0$$

$$I_3 = \int_{-l}^{l} \left(\sum_{n=1}^{\infty} b_n \sin kx\right)^2 dx = \int_{-l}^{l} \left(b_1 \sin\frac{\pi}{l}x + b_2 \sin\frac{2\pi}{l}x + \cdots\right)\left(b_1 \sin\frac{\pi}{l}x + b_2 \sin\frac{2\pi}{l}x + \cdots\right) dx$$

$$= \underbrace{\sum_{n=1}^{\infty} \int_{-l}^{l} b_n^2 \sin^2\frac{n\pi}{l}x \, dx}_{n=m} + \sum_{n=1}^{\infty} b_n b_m \sin\frac{n\pi}{l}x \sin\frac{m\pi}{l}x \, dx = \sum_{n=1}^{\infty} l\, b_n^2 + 0$$

$$\Rightarrow \int_{-l}^{l} f^2(x)\, dx = A + B + C = \frac{a_0^2}{2} l + 0 + \sum_{n=1}^{\infty} l\left(a_n^2 + b_n^2\right) = l\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right)$$

ثابت کنید که (الف)

$$\ln\left(2\sin\frac{x}{2}\right) = -\sum_{n=1}^{\infty} \frac{\cos nx}{n} \quad ; \quad 0 < x < \pi$$

…به بسط $\ln$ از ریاضیات مهندسی ۱ داریم:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

$$z = -e^{ix} \longrightarrow \ln(1-e^{ix}) = -e^{ix} - \frac{(-e^{ix})^2}{2} + \frac{(-e^{ix})^3}{3} - \dots$$

$$\ln(1-e^{ix}) = \sum_{n=1}^{\infty} -\frac{1}{n}(\cos nx + i\sin nx)$$

$$2\sin\frac{x}{2} = 2\left(\frac{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}{2i}\right) = i\,e^{-\frac{ix}{2}}(1-e^{ix})$$

$$\ln\left(2\sin\frac{x}{2}\right) = \ln\left(i\,e^{-\frac{ix}{2}}(1-e^{ix})\right)$$

$$\mathrm{Re}\left[\ln i e^{-\frac{ix}{2}}(1-e^{ix})\right] = \mathrm{Re}\left(\ln i e^{-\frac{ix}{2}} + \ln(1-e^{ix})\right) = \ln(1-e^{ix})$$

$$\ln\left(2\sin\frac{x}{2}\right) = -\sum_{n=1}^{\infty} \frac{\cos nx}{n} \qquad 0 < x < \pi$$

ب)

$$\ln\left(2\cos\frac{x}{2}\right) = \sum_{n=1}^{\infty} -(-1)^{n+1}\frac{\cos nx}{n} \quad ; \quad -\pi < x < \pi$$

اگر در بسط $\ln$ بالا به جای z قرار دهیم e^{ix} خواهیم داشت:

$$\ln(1+e^{ix}) = e^{ix} - \frac{(e^{ix})^2}{2} + \frac{(e^{ix})^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{(ix)^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(\cos nx + i\sin nx)$$

$$2\cos\frac{x}{2} = 2\left(\frac{e^{\frac{ix}{2}} + e^{-\frac{ix}{2}}}{2}\right) = (e^{ix}+1)e^{-\frac{ix}{2}}$$

$$\mathrm{Re}\left\{\ln\left(e^{\frac{ix}{2}}(1+e^{ix})\right)\right\} = \frac{ix}{2} + \ln(1+e^{ix}) = \ln(1+e^{ix})$$

۱۴) انتگرال فوریه هریک از توابع را بیابید :

$$a)\quad f(x)=\begin{cases} x & ; \quad 0<x<a \\ 0 & ; \quad x>a \end{cases} \qquad f(-x)=f(x)$$

تابع زوج است بنابراین داریم : $B(w)=0$

$$A(w)=\frac{2}{\pi}\int_0^a x\cos wx\,dx=\frac{2}{\pi}\left(\frac{x}{w}\sin wx+\frac{1}{w^2}\cos wx\right)_0^a$$

$$=\frac{2}{\pi}\left(\frac{a}{w}\sin aw+\frac{1}{w^2}\cos aw-\frac{1}{w^2}\right)$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\left[\frac{a\sin aw}{w}+\frac{\cos aw-1}{w^2}\right]\cos wx\,dw$$

$$b)\quad f(x)=\begin{cases} e^{-x}+e^{-2x} & x>0 \\ f(-x) & x<0 \end{cases}$$

تابع زوج است $B(w)=0$

$$A(w)=\frac{1}{\pi}\int_0^\infty\left(e^{-x}+e^{-2x}\right)\cos wx\,dx=\frac{1}{\pi}\int_0^\infty e^{-x}\cos wx\,dx+\frac{1}{\pi}\int_0^\infty e^{-2x}\cos wx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^{-x}}{1+w^2}\left(-\cos wx+w\sin wx\right)+\frac{e^{-2x}}{4+w^2}\left(-2\cos wx+w\sin wx\right)\right]_0^\infty$$

$$=\frac{1}{\pi}\left[(0+0)-\left(\frac{1}{1+w^2}\times(-1)+\frac{1}{4+w^2}\times(-2)\right)\right]=\frac{3}{\pi}\times\frac{2+w^2}{4+5w^2+w^4}$$

$$f(x)=\frac{3}{\pi}\int_0^\infty\frac{2+w^2}{4+5w^2+w^4}\cos wx\,dx$$

$$c)\quad f(x)=\begin{cases} x^2 & ; \quad 0<x<a \\ 0 & ; \quad x>a \end{cases} \quad ,\ f(x)=f(-x)$$

تابع زوج است $B(w)=0$

$$A(w)=\frac{1}{\pi}\int_0^a x^2\cos wx\,dx=\frac{1}{\pi}\left[\frac{x^2}{w}\sin wx+\frac{2x\cos wx}{w^2}-\frac{2}{w^3}\sin wx\right]_0^a$$

$$=\frac{1}{\pi}\left[\frac{a^2}{w}\sin aw+\frac{2a}{w^2}-\frac{2}{w^3}\sin aw-0\right]$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\left[\left(a^2-\frac{2}{w^2}\right)\sin aw+\frac{2a}{w}\cos aw\right]\frac{\cos wx}{w}\,dw$$

الف) هرگاه $f(-x)=f(x)$ آنگاه ثابت کنید:

a) $f(bx)=\frac{1}{b}\int_0^\infty a\left(\frac{w}{b}\right)\cos wx\,dw \quad ; \quad b>0$

$$f(u)=\int_0^\infty a'(w)\cos wu\,dw \quad , \quad a'(w)=\frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du=a(w)$$

$$f(u)=\int_0^\infty a(w)\cos wu\,dw \xrightarrow{u=bx} f(bx)=\int_0^\infty a(w)\cos wbx\,dw$$

$$wb=w' \Rightarrow b\,dw=dw'$$

$$\Rightarrow f(bx)=\int_0^\infty a\left(\frac{w'}{b}\right)\cos w'x\,\frac{dw'}{b}=\frac{1}{b}\int_0^\infty a\left(\frac{w'}{b}\right)\cos w'x\,dw'$$

---

b) $x^2 f(x)=\int_0^\infty a^*(w)\cos wx\,dw \quad , \quad a^*=-\frac{d^2 a}{dw^2}$

$$\int_0^\infty a^*(w)\cos wx\,dw=\int_0^\infty -\frac{d^2}{dw^2}a(w)\cos wx\,dw=-\int_0^\infty a''(w)\cos wx\,dw$$

با استفاده از جزء به جزء داریم: $\int_0^\infty a''(w)\cos wx\,dw=a'(w)\cos wx\Big]_0^\infty-\int_0^\infty(-x\sin wx)\,a'(w)\,dw$

$$* \Rightarrow \int_0^\infty a^*(w)\cos wx\,dw=-\left[\left[a'(w)\cos wx\right]_0^\infty+\;a(w)\,x\sin wx\Big]_0^\infty-\int_0^\infty x^2 a(w)\cos wx\,dw\right]$$

$$=\left[-a'(w)\cos wx\right]_0^\infty-\left[a(w)\,x\sin wx\right]_0^\infty+x^2\int_0^\infty a(w)\cos wx\,dw=I_1+I_2+x^2 f$$

باید نشان دهیم I_1 و I_2 هر دو صفرند

$$I_1=\left[-a'(w)\cos wx\right]_0^\infty=0 \quad , \quad I_2=0$$

راه حل دوم مورد b

$$a^*(w)=\frac{2}{\pi}\int_0^\infty x^2 f(x)\cos wx\,dx \qquad *$$

$$a(w)=\frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx \xrightarrow[\text{نسبت به } w]{\text{با مشتق}} a'(w)=-\frac{2}{\pi}\int_0^\infty x f(x)\sin wx\,dx \xrightarrow[\text{نسبت به } w]{\text{مشتق گیری}}$$

$$a''(w)=-\frac{2}{\pi}\int_0^\infty x^2 f(x)\cos wx\,dx \xrightarrow{\;*\;} a''(w)=-a^*(w) \Rightarrow a^*(w)=-\frac{d^2 a}{dw^2}$$

۱۴- ثابت کنید که $f(x)=-f(-x)\ ;\ x<0\ ,\ x>0\ ;\ $ $\quad$ a) $\displaystyle\int_0^{\infty} \frac{w^3 \sin wx}{w^4+4}\,dw = \frac{\pi}{2} e^{-x}\cos x$

$$\text{if}\quad f(-x)=-f(x) \Rightarrow a(w)=0 \quad , f(x)=\int_0^{\infty} b(w)\sin wx\,dw$$

فرض: $\displaystyle b(w)=\frac{w^3}{w^4+4} \quad , \int_0^{\infty} b(w)\sin wx\,dw = f(x) \Rightarrow b(w)=\frac{2}{\pi}\int_0^{\infty} f(x)\sin wx\,dx$

$$b(w)=\frac{2}{\pi}\int_0^{\infty}\frac{\pi}{2}e^{-x}\cos x\cdot\sin wx\,dx=\int_0^{\infty}e^{-x}\cos x\cdot\sin wx\,dx=\frac{1}{2}\int_0^{\infty}e^{-x}\left(\sin(w+1)x+\sin(w-1)x\right)$$

$$\Rightarrow 2b(w)=\int_0^{\infty}e^{-x}\sin(w+1)x\,dx+\int_0^{\infty}e^{-x}\sin(w-1)x\,dx$$

با توجه به رابطه‌ی * در پشت سوال ۴ داریم:

$$2b(w)=\frac{e^{-x}}{1+(w+1)^2}\left(-\sin(w+1)x-(w+1)\cos(w+1)x\right)\Big]_0^{\infty}+$$

$$\frac{e^{-x}}{1+(w-1)^2}\left[-\sin(w-1)x-(w-1)\cos(w-1)x\right]_0^{\infty}$$

$$\Rightarrow 2b(w)=\frac{w+1}{1+(w+1)^2}+\frac{w-1}{1+(w-1)^2}=\frac{2w^3}{w^4+4}\Rightarrow b(w)=\frac{w^3}{w^4+4}$$

$$\text{b)}\quad \int_0^{\infty}\frac{\cos(w\frac{\pi}{2})\cos wx}{1-w^2}\,dw=\begin{cases}\frac{\pi}{2}\cos x & ; |x|<\frac{\pi}{2}\\[2ex] 0 & ; |x|>\frac{\pi}{2}\end{cases}$$

$$\text{if}\quad f(-x)=f(x)\Rightarrow b(w)=0\quad , f(x)=\int_0^{\infty}a(w)\cos wx\,dw$$

$$\int_0^{\infty}\frac{\cos(\frac{\pi}{2}w)}{1-w^2}\cos wx\,dw=\int_0^{\infty}a(w)\cos wx\,dw=f(x)\Rightarrow a(w)=\frac{2}{\pi}\int_0^{\infty}f(x)\cos wx\,dx$$

هدف از برهان این است که اگر $f(x)$ سمت راست b باشد آنگاه $a(w)$: $\quad \displaystyle a(w)=\frac{\cos\frac{\pi}{2}w}{1-w^2}$

$$a(w)=\frac{2}{\pi}\int_0^{\infty}f(x)\cos wx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\pi}{2}\cos x\cos wx\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\cos(w+1)x+\cos(w-1)x\right)dx$$

$$=\frac{1}{2(w+1)}\sin(w+1)x\Big]_0^{\frac{\pi}{2}}+\frac{1}{2(w-1)}\sin(w-1)x\Big]_0^{\frac{\pi}{2}}=\frac{\sin(w+1)\frac{\pi}{2}}{2(w+1)}+\frac{\sin(w-1)\frac{\pi}{2}}{2(w-1)}$$

$$=\frac{-2\cos w\frac{\pi}{2}}{2(w^2-1)}=\frac{\cos w\frac{\pi}{2}}{1-w^2}$$

$$c)\quad \int_0^\infty \frac{\sin w \cos wx}{w}\,dw = \begin{cases} \frac{\pi}{2} & ; \ 0<x<1 \\ \frac{\pi}{4} & x=1 \\ 0 & x>1 \end{cases}$$

$f(-x)=f(x) \qquad -\infty<x<0$

چون f زوج است $\Leftarrow b(w)=0$

$$a(w)=\frac{2}{\pi}\int_0^1 \frac{\pi}{2}\cos wx\,dx + \frac{2}{\pi}\int_1^\infty 0\cos wx\,dx = \frac{1}{\pi}\int_0^1 \pi\cos wx\,dx = \frac{\sin w}{w}$$

$$\text{if } x=1 \Rightarrow \int_0^\infty \frac{\sin w\cos w}{w}\,dw = \int_0^\infty \frac{\sin 2w}{2w}\,dw = \frac{1}{2}\int_0^\infty \frac{\sin u}{u}\,du = \frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

۱۷- سری فوریه مختلط هر یک از توابع زیر را بیابید به کمک آن سری فوریه حقیقی متناظر با آنها را معین کنید.

$$a)\ f(x)=e^{2x} \quad -\pi<x<\pi \quad ; \quad f(x)=\sum_{-\infty}^{+\infty}\frac{1}{\pi}\,\frac{2+in}{4+n^2}(-1)^n\sinh 2\pi\, e^{inx}$$:جواب

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(2-in)x}\,dx = \frac{1}{2\pi}\times\frac{1}{2-in}\left(e^{(2-in)\pi}-e^{-(2-in)\pi}\right)$$

$$e^{in\pi}=(-1)^n=\cos n\pi+\underbrace{\cancel{i\sin n\pi}}_{0} \Rightarrow c_n=\frac{1}{2\pi}\times\frac{1}{2-in}\times\frac{2+in}{2+in}\left(e^{2\pi}e^{-in\pi}-e^{-2\pi}e^{+in\pi}\right)$$

$$=\frac{1}{2\pi}\times\frac{2+in}{4+n^2}\times 2\times(-1)^n\sinh 2\pi$$

$$f(x)=\sum_{-\infty}^{+\infty}\frac{1}{\pi}\,\frac{2+in}{4+n^2}(-1)^n\sinh 2\pi\, e^{inx}$$

$a_n=c_n+c_{-n}$

$b_n=i(c_n-c_{-n})$

$$b)\ f(x)=x \quad ; \quad -\pi<x<\pi \quad \rightarrow c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,e^{-inx}\,dx$$

$$=\frac{1}{2\pi}\left(-\frac{x}{in}+\frac{1}{n^2}\right)e^{-inx}\Big]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\left(-\frac{\pi}{in}+\frac{1}{n^2}\right)e^{-in\pi}-\left(\frac{\pi}{in}+\frac{1}{n^2}\right)e^{in\pi}\right]$$

$$=-\frac{\pi}{in}\left(e^{in\pi}+e^{-in\pi}\right)-\frac{1}{n^2}\underbrace{\left(e^{in\pi}-e^{-in\pi}\right)}_{0}=\frac{i}{n}(-1)^n$$

$$f(x)=\sum_{-\infty}^{+\infty}(-1)^n\frac{i}{n}e^{inx}$$

۱۸-

a)
$$f(x)=\begin{cases} e^{-2x} & ; \quad -\pi<x<\pi \\ 0 & ; \quad \text{جاهای دیگر} \end{cases}$$

$$c(w)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2x}e^{-iwx}\,dx=\frac{1}{2\pi}\left[\frac{-e^{-(2x+iwx)}}{(2+iw)}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\left[\frac{e^{(2+iw)\pi}-e^{-(2+iw)\pi}}{2+iw}\right]=\frac{2-iw}{\pi(4+w^2)}\left[\frac{e^{(2+iw)\pi}-e^{-(2+iw)\pi}}{2}\right]$$

$$f(x)=\int_{-\infty}^{+\infty}\frac{2-iw}{\pi(4+w^2)}\left[\frac{e^{(2+iw)\pi}-e^{-(2+iw)\pi}}{2}\right]e^{iwx}\,dw$$

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b)
$$f(x)=\begin{cases} \sinh 2x & ; \quad 0<x<2 \\ 0 & ; \quad \text{جاهای دیگر} \end{cases}$$

$$c(w)=\frac{1}{2\pi}\int_0^2 \sinh 2x\, e^{-iwx}\,dx=\frac{1}{4\pi}\int_0^2 e^{2x}e^{-iwx}\,dx-\frac{1}{4\pi}\int_0^2 e^{-2x}e^{-iwx}\,dx$$

$$=\frac{1}{4\pi}\left[\frac{e^{(2-iw)x}}{(2-iw)}+\frac{e^{-(2+iw)x}}{(2+iw)}\right]_0^2=\frac{1}{4\pi}\left[\frac{e^{4-2iw}}{2-iw}+\frac{e^{-4-2iw}}{2+iw}-\frac{1}{2-iw}-\frac{1}{2+iw}\right]$$

$$f(x)=\int_{-\infty}^{+\infty}c(w)\,e^{iwx}\,dw$$

۱- آیا بسط‌های کسینوسی و سینوسی سری فوریه تابع $f(x)=e^{x}$ موجود است.

خیر، زیرا e^{x} به طور مطلق انتگرال‌پذیر نیست $\left|\int_{0}^{\infty} e^{x}dx\right| > \infty$

۲- $F_s\{e^{-ax}\}$; $a>0$ را با انتگرال‌گیری به دست آورید:

$$F_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin wx\,dx$$

حاصل انتگرال $\int_0^\infty e^{-ax}\sin wx\,dx$ به روش جزء به جزء قابل محاسبه است. بدین ترتیب که

$$I_0=\int_0^\infty e^{-ax}\sin wx\,dx = uv-\int_0^\infty v\,du = -\frac{e^{-ax}\cos wx}{w} - \underbrace{\int_0^\infty \frac{a}{w}e^{-ax}\cos wx\,dx}_{I}$$

$$u=e^{-ax} \Rightarrow du=-ae^{-ax}dx$$

$$dv=\sin wx\,dx \Rightarrow v=-\frac{\cos wx}{w}$$

برای I باز هم به روش جزء به جزء خواهیم داشت:

$$I=\frac{a}{w}\int_0^\infty e^{-ax}\cos wx\,dx = uv-\int_0^\infty v\,du = \frac{a}{w^2}e^{-ax}\sin wx+\frac{a^2}{w^2}\underbrace{\int_0^\infty e^{-ax}\sin wx\,dx}_{I_0}$$

$$u=e^{-ax} \Rightarrow du=-ae^{-ax}dx$$

$$dv=\cos wx\,dx \Rightarrow v=\frac{\sin wx}{w}$$

$$\Rightarrow I_0=-\frac{1}{w}e^{-ax}\cos wx-\frac{a}{w^2}e^{-ax}\sin wx-\frac{a^2}{w^2}\underbrace{\int_0^\infty e^{-ax}\sin wx\,dx}_{I_0}$$

$$\Rightarrow I_0=\left[-\frac{1}{w}\left(e^{-ax}\cos wx+\frac{a}{w}e^{-ax}\sin wx\right)\frac{w^2}{a^2+w^2}\right]_0^\infty$$

$$\Rightarrow I_0=-\frac{w}{a^2+w^2}$$

$$\Rightarrow F_s\{e^{-ax}\}=\sqrt{\frac{2}{\pi}}\times\frac{-w}{a^2+w^2}$$

۲۲- تبدیل کسینوسی وارون تابع e^{-w} را بیابید.

$$F_c^{-1}\{f\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-w} \cos wx \, dw$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a\cos bx + b\sin bx) + c$$

در اینجا $x=b$ و $a=-1$:

$$F_c^{-1}\{f(w)\} = \sqrt{\frac{2}{\pi}} \; \frac{e^{-w}}{1+x^2} \left(-\cos wx + x\sin wx\right)\Big]_0^\infty = \sqrt{\frac{2}{\pi}} \left(0 - \frac{e^0}{1+x^2}(-1)\right) = \sqrt{\frac{2}{\pi}} \frac{1}{1+x^2}$$

۲۳- تبدیل سینوسی وارون $\hat{f}(w)$ را بیابید.

$$F_s^{-1}\{\hat{f}(w)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(w) \sin wx \, dx$$

$$F_s^{-1}\{\hat{f}(w)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{1}{w} - \frac{\cos w\pi}{w}\right) \sin wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin wx}{w} dw - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin wx \cos w\pi}{w} dw$$

از آنجا که $\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$ پس انتگرال روبرو قابل حل است.

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin wx}{wx} dw - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{2w} \left(\sin (x+\pi)w + \sin (x-\pi) w\right) dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \frac{\sin wx}{wx} dw - \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \int_0^\infty (x+\pi) \frac{\sin (x+\pi)w}{(x+\pi)w} + \frac{\sin (x-\pi)w}{(x-\pi)(w)} (x-\pi) \; dw$$

$$= \sqrt{\frac{2}{\pi}} \left[x \times \frac{\pi}{2} - \frac{(x+\pi)}{2} \times \frac{\pi}{2} - \frac{(x-\pi)}{2} \times \frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2}} (x-\pi)$$

$$\Rightarrow F^{-1}\{\hat{f}(w)\} = \sqrt{\frac{\pi}{2}} (x-\pi)$$

۲ـ آیا تبدیل کسینوسی فوریه تابع $\frac{\cos x}{x}$ یا $\frac{\sin x}{x}$ موجود است؟

الف: $F_c\left\{\frac{\cos x}{x}\right\}$ موجود نیست زیرا $\frac{\cos x}{x}$ به طور مطلق انتگرال پذیر نیست، چون در همسایگی راست صفر مانند $\frac{1}{x}$ عمل می‌کند و $\int_0^a \frac{1}{x}dx \quad a>0$ واگراست.

ب: $F_c\left\{\frac{\sin x}{x}\right\}$ موجود است، زیرا $\frac{\sin x}{x}$ مطلقاً انتگرال پذیر و هموار است.

$$F_c\left\{\frac{\sin x}{x}\right\}=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x \cdot \cos wx}{x}dx$$

که انتگرال مقابل در سؤال ۲۸ محاسبه خواهد شد.

۲۵ـ تبدیل فوریه هریک از توابع زیر را بدون استفاده از جدول تبدیلات فوریه به دست آورید.

$$a)\quad f(x)=\begin{cases} e^{-x} & ; \quad x>0 \\ 0 & ; \quad x<0\end{cases}$$

$$F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(iw+1)x}dx=\frac{1}{\sqrt{2\pi}(1+iw)}e^{-(1+iw)x}\Big]$$

$$\left[=\frac{-1}{\sqrt{2\pi}(1+iw)}(0-1)=\frac{1}{\sqrt{2\pi}(1+iw)}\right]$$

$$b)\quad f(x)=\begin{cases} e^{x} & ; \quad x>0 \\ 0 & ; \quad x<0\end{cases}$$

$$F\{f\}=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{x}\cdot e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\frac{1}{(1-iw)}e^{(1-iw)x}\Big]_0^\infty=\frac{1}{\sqrt{2\pi}}\frac{e^{-(iw-1)x}}{1-iw}\Big]_0^\infty$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{(1-iw)}(0-1)=\frac{1}{\sqrt{2\pi}(iw-1)}$$

c) $f(x) = \begin{cases} e^{2ix} & ; \quad |x| < 1 \\ 0 & ; \quad |x| \geq 1 \end{cases}$

$$F\{f\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(0 + \int_{-1}^{1} e^{2ix} e^{-i\omega x} dx\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i(2-\omega)x} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i(2-\omega)} e^{i(2-\omega)x} \Big]_{-1}^{1} = \frac{1}{\sqrt{2\pi}\; i(2-\omega)} \left(e^{i(2-\omega)} - e^{-i(2-\omega)}\right)$$

$$= \frac{2\sin(2-\omega)}{\sqrt{2\pi}\;(2-\omega)}$$

توجه: * $\left[\left(e^{i(2-\omega)} - e^{-i(2-\omega)}\right) = 2i\sin(2-\omega)\right]$

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d) $f(x) = \begin{cases} x & ; \quad 0 < x < a \\ 0 & ; \quad \text{جاهای دیگر} \end{cases}$

$$F\{f\} = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} x e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[-\frac{x}{i\omega} - \frac{1}{(i\omega)^2}\right] e^{-i\omega x} \Big]_{0}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[-\frac{a}{i\omega} - \frac{1}{(i\omega)^2}\right] e^{-i\omega a} + \frac{1}{(i\omega)^2}\right]$$

۲۶- نشان دهید اگر $f(x)$ دارای تبدیل فوریه باشد $f(x-a)$ نیز دارای تبدیل فوریه است و $F\{f(x-a)\} = e^{-i\omega a} F\{f(x)\}$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx \quad \text{و} \quad F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-a) e^{-i\omega(x-a)} d(x-a)$$

$u = x - a \qquad x = u + a \qquad dx = du$

$$F\{f(u)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) e^{-i\omega u} e^{-i\omega a} du = \frac{e^{-i\omega a}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) e^{-i\omega u} du$$

$u \to x$

$$\Rightarrow F\{f(x-a)\} = \frac{e^{-i\omega a}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx = e^{-i\omega a} F\{f(x)\}$$

۲۷ـ نشان دهید که اگر $\hat{f}(w)$ تبدیل فوریه f باشد آنگاه $\hat{f}(w-a)$ تبدیل فوریه $f(x)\,e^{iax}$ است.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{-iwx}\,dx \quad , \quad \hat{f}(w-a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{-i(w-a)x}\,dx$$

$$\Rightarrow \hat{f}(w-a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{-iwx}\,e^{iax}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \left(f(x)\,e^{iax}\right) e^{-iwx}\,dx = F\left\{f(x)\,e^{iax}\right\}$$

۲۸ـ انتگرال فوریه تابع $f(x)=\begin{cases}1 & ; |x|<1\\ 0 & ; |x|>1\end{cases}$ را بیابید و به کمک آن انتگرال $\int_0^\infty \frac{\sin x \cdot \cos ax}{x}\,dx$ را به ازای مقادیر گوناگون a بیابید.

$$f(-x)=f(x) \Rightarrow b(w)=0$$

$$f(x)=\int_0^\infty a(w)\cos wx\,dw \quad , \quad a(w)=\frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx$$

$$a(w)=\frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx = \frac{2}{\pi}\int_0^1 \cos wx\,dx + 0 = \frac{2}{\pi w}\sin wx\Big]_0^1 = \frac{2}{\pi w}\sin w$$

$$\Rightarrow f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\,dx \Rightarrow \int_0^\infty \frac{\sin w\,\cos wx}{w}\,dx = \frac{\pi}{2}f(x)$$

$$\Rightarrow \int_0^\infty \frac{\sin x\cdot\cos ax}{x}\,dx = \frac{\pi}{2}f(a) = \begin{cases}\frac{\pi}{2} & ; |a|<1\\ 0 & ; |a|>1\end{cases}$$

if $|a|=1 \Rightarrow$

1) $a=1 \quad \int_0^\infty \frac{\sin x\cdot\cos x}{x}\,dx = \int_0^\infty \frac{\sin 2x}{2x}\,dx = \frac{\pi}{4}$

2) $a=-1 \quad \int_0^\infty \frac{\sin x\cdot\cos(-x)}{x}\,dx = \frac{\pi}{4}$

۲۹- به کمک انتگرال فوریه نشان دهید:

a) $\int_0^\infty \frac{x\sin ax}{1+x^2}\,dx = \frac{\pi}{2}e^{-a} \qquad a>0$

$\int_0^\infty \frac{w\sin xw}{1+w^2}\,dw = \frac{\pi}{2}e^{-x} \qquad x>0 \qquad \int_0^\infty b(w)\sin wx\,dw = \frac{\pi}{2}e^{-x}$

$\frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-x}\sin wx\,dx = \int_0^\infty e^{-x}\left(\frac{e^{iwx}-e^{-iwx}}{2i}\right)dx$

$= \frac{1}{2i}\int_0^\infty \left(e^{-x(1-iw)} - e^{-x(1+iw)}\right)dx = \frac{1}{2i}\frac{1}{iw-1}e^{-x(1-iw)}\Big]_0^\infty + \frac{1}{2i}\times\frac{1}{1+iw}e^{-x(1+iw)}\Big]_0^\infty$

$= \frac{1}{2i}\frac{1}{iw-1} - \frac{1}{2i}\frac{1}{iw+1} = \frac{1}{2i}\left(\frac{-1}{iw-1}+\frac{-1}{iw+1}\right) = \frac{-2iw}{2i(i^2w^2-1)} = \frac{w}{1+w^2} = b(w)$

b) $\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \frac{\pi}{2}e^{-a} \qquad a\geq 0$

$\int_0^\infty \frac{\cos wx}{1+w^2}\,dw = \frac{\pi}{2}e^{-x}$

$a(w) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-x}\cos wx\,dx = \int_0^\infty e^{-x}\frac{(e^{iwx}+e^{-iwx})}{2}\,dx$

$= \frac{1}{2}\int_0^\infty \left(e^{-x(1-iw)} + e^{-x(1+iw)}\right)dx$

$= \frac{1}{2}\frac{1}{iw-1}e^{-x(1-iw)} - \frac{1}{2}\frac{e^{-x(1+iw)}}{(1+iw)}e^{-x(1+iw)}\Big]_0^\infty$

$= \frac{1}{2}\frac{1}{1-iw} + \frac{1}{2}\frac{1}{1+iw} = \frac{2}{2}\frac{1}{1+w^2} = \frac{1}{1+w^2}$

ـ تبدیلات فوریه مختلط از توابع زیر را بیابید:

a) $f(x)=\begin{cases} 1 & ; \ |x|<\pi \\ 0 & \ |x|>\pi \end{cases}$

$$F\{f\}=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i\omega x}\,dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-\pi}^{\pi}=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i\omega\pi}-e^{-i\omega\pi}}{i\omega}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\ \frac{2\sin\omega\pi}{\omega}=\frac{2}{\sqrt{2\pi}}\ \frac{\sin\omega\pi}{\omega}$$

---

b) $f(x)=\begin{cases} x^2 & ; \ |x|<x_0 \\ 0 & ; \ |x|>x_0 \end{cases}$

$$F\{f\}=\frac{1}{\sqrt{2\pi}}\int_{-x_0}^{x_0} x^2 e^{-i\omega x}\,dx=\frac{1}{\sqrt{2\pi}}\left[\frac{-x^2 e^{-i\omega x}}{i\omega}+\frac{2x e^{-i\omega x}}{\omega^2}+\frac{2e^{-i\omega x}}{i\omega^3}\right]_{-x_0}^{x_0}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{-x_0^2 e^{-i\omega x_0}}{i\omega}+\frac{2x_0 e^{-i\omega x_0}}{\omega^2}+\frac{2e^{-i\omega x_0}}{i\omega^3}+\frac{x_0^2 e^{i\omega x_0}}{i\omega}+\frac{2x_0 e^{i\omega x_0}}{\omega^2}-\frac{2e^{i\omega x_0}}{i\omega^3}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2}{\omega}\left[x_0^2-\frac{2}{\omega^2}\right]\sin\omega x_0+\frac{4x_0}{\omega^2}\cos\omega x_0\right]$$

۳۱ـ تبدیل فوریه تابع $f(x)=\begin{cases} 1-x^2 & ; \ |x|<1 \\ 0 & ; \ |x|>1 \end{cases}$ را بدست آورید و به کمک آن $\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}\,dx$ را محاسبه کنید.

$$\sqrt{2\pi}\ F\{f\}=\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\,dx=\int_{-1}^{1}(1-x^2)e^{-i\omega x}\,dx$$

$$\Rightarrow \sqrt{2\pi}\ F\{f\}=\left[\left(-\frac{1}{i\omega}(1-x^2)+\frac{2x}{(i\omega)^2}+\frac{2}{(i\omega)^3}\right)e^{-i\omega x}\right]_{-1}^{1}$$

(ادامه در صفحه بعد)

ادامه سوال ۳۱.

$$= \left(\frac{2}{(iw)^2} + \frac{2}{(iw)^3}\right)e^{-iw} - \left(\frac{2}{(iw)^3} - \frac{2}{(iw)^2}\right)e^{iw}$$

$$= \frac{2}{(iw)^2}\left(e^{-iw}+e^{iw}\right) - \frac{2}{(iw)^3}\left(e^{+iw}-e^{-iw}\right) = \frac{4}{(iw)^2}\cos w - \frac{4}{(iw)^3}\, i\sin w = -\frac{4\cos w}{w^2} + \frac{4}{w^3}\sin w$$

$$\Rightarrow \hat{F}(w) = -\frac{4}{\sqrt{2\pi}}\ \frac{w\cos w - \sin w}{w^3}$$

$$, \ f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{F}(w)e^{iwx}\,dx = \frac{-4}{2\pi}\int_{-\infty}^{+\infty}\frac{w\cos w - \sin w}{w^3}\,e^{iwx}\,dw$$

$$\Rightarrow f(0) = -\frac{4}{2\pi}\int_{-\infty}^{+\infty}\frac{w\cos w - \sin w}{w^3}\,dw = 1 \Rightarrow \int_{-\infty}^{+\infty}\frac{w\cos w - \sin w}{w^3}\,dw = -\frac{\pi}{2}$$

$$g(w) = \frac{w\cos w - \sin w}{w^3} \quad , \ g(-w) = \frac{-w\cos(-w) - \sin(-w)}{(-w)^3} = g(w)$$

$$\Rightarrow \int_{-\infty}^{+\infty} g(w)\,dw = -\frac{\pi}{2} \Rightarrow \int_0^{\infty} g(w)\,dw = \frac{1}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

۳۲- تبدیلات فوریه کسینوسی و سینوسی توابع زیر را بیابید.

a) $$f(x) = \begin{cases} 1 & ; \ 0 \leq x < 1 \\ 0 & ; \ x \geq 1 \end{cases}$$

$$F_c\{f\} = \sqrt{\frac{2}{\pi}}\int_0^1 1\times\cos wx\,dx = \sqrt{\frac{2}{\pi}}\times\frac{1}{w}\sin wx\Big]_0^1 = \sqrt{\frac{2}{\pi}}\ \frac{\sin w}{w}$$

$$F_s\{f\} = \sqrt{\frac{2}{\pi}}\int_0^1 \sin wx\,dx = -\frac{1}{w}\sqrt{\frac{2}{\pi}}\cos wx\Big]_0^1 = \sqrt{\frac{2}{\pi}}\ \frac{1-\cos w}{w}$$

---

b) $f(x) = e^{-ax}\ (a>0)$ $$F_c\{f\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos wx\,dx = \sqrt{\frac{2}{\pi}}\ \frac{e^{-ax}}{a^2+w^2}\left(-a\cos wx + w\sin wx\right)\Big|_0^{\infty}$$

$$\Rightarrow F_c\{f\} = \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+w^2}$$

$$F_s\{f\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin wx\,dx = \sqrt{\frac{2}{\pi}}\times\frac{e^{-ax}}{a^2+w^2}\left(-a\sin wx - w\cos wx\right)\Big|_0^{\infty}$$

$$\Rightarrow F_s\{f\} = \sqrt{\frac{2}{\pi}}\left(0 - \frac{1}{a^2+w^2}(-w)\right) = \sqrt{\frac{2}{\pi}}\left(\frac{w}{a^2+w^2}\right)$$

۳۳ـ تبدیل فوریه سینوسی تابع $e^{-|x|}$ را بدست آورید و به کمک آن $\int_0^\infty \frac{x\sin mx}{1+x^2}dx$ را محاسبه کنید.

$$F_s\left\{f(x)\right\}=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\sin wx\,dx \qquad 0\leqslant x\leqslant\infty \Rightarrow |x|=x \Rightarrow F_s\{f\}=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin wx\,dx$$

$$2i\sin wx=e^{iwx}-e^{-iwx} \Rightarrow e^{-x}\sin wx=\frac{1}{2i}\left(e^{(iw-1)x}-e^{-(iw+1)x}\right)$$

$$F_s\{f\}=\frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[\frac{-1}{(1-iw)}e^{-(1-iw)x}+\frac{1}{(iw+1)}e^{-(iw+1)x}\right]_0^\infty=\frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[0-\left(\frac{-1}{1-iw}+\frac{1}{iw+1}\right)\right]$$

$$\Rightarrow F_s\{f\}=\sqrt{\frac{2}{\pi}}\;\frac{w}{1+w^2}$$

$$F_s^{-1}\{f\}=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(w)\sin wx\,dw=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{w}{1+w^2}\sin wx\,dw=e^{-|x|}$$

اگر $x=m \Rightarrow \int_0^\infty \frac{w\sin mw}{1+w^2}dw=\frac{\pi}{2}e^{-|m|}$ و (با تغییر متغیر از w به x نتیجه حاصل می‌شود)

۳۴ـ چنانچه $f(x)$ تابعی پیوسته تکه‌ای باشد در فاصله $[-\pi,\pi]$ و متناوب با دوره تناوب 2π نیز باشد، آنگاه ثابت کنید توابع زیر نیز دارای چنین خاصیتی هستند.

$$g(x)=f(w+x)\;\frac{\sin\left\{(n+\frac{1}{2})(w+x)\right\}}{2\sin\frac{w}{2}}\quad ,\ n=1,2,3,\ldots$$

برهان:
$$\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}=\frac{2i\sin(n+\frac{1}{2})x}{2i\sin\frac{x}{2}}=\frac{e^{i(n+\frac{1}{2})x}-e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}=$$

$$\frac{e^{-i\frac{x}{2}}\left(e^{i(n+1)x}-e^{-inx}\right)}{e^{-i\frac{x}{2}}\left(e^{ix}-1\right)}=\frac{e^{i(n+1)x}-e^{-inx}}{e^{ix}-1}=\frac{\sum_{k=-n}^{n}e^{i(k+1)x}-e^{ikx}}{e^{ix}-1}$$

$$=\frac{\sum_{k=-n}^{n}e^{ikx}\left(e^{ix}-1\right)}{e^{ix}-1}=\sum_{k=-n}^{n}e^{ikx}=D_n(x)=1+\sum_{k=1}^{n}e^{ikx}+e^{-ikx}=1+\sum_{k=1}^{n}2\cos kx$$

$D_n(x)$ تابعی پیوسته و متناوب است.

$$\Rightarrow D_n(x)=\sum_{k=-n}^{n}e^{ikx}=1+2\sum_{k=1}^{n}\cos kx$$

$$g(x) = f(x+w)\, D_n(x+w)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin\frac{w}{2}}$$

* دراین برهان ایشان دهیم : ۱) و پیوسته ... است . ۲) متناوب است.

1) $g(x)= f(x+w)\, D_n(w+x)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin\frac{w}{2}}$, $g(x-w) = f(x)\,.\,D_n(x)\ \frac{\sin x/2}{2\sin\frac{w}{2}}$

$$\lim g(x) - g(x-w) = \lim\ f(x+w)\, D_n(x+w)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin\frac{w}{2}} - f(x)\, D_n(x)\ \frac{\sin x/2}{2\sin w/2}$$

$$= \lim_{w\to 0^-} \Big[f(x+w)\, D_n(x+w)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin w/2} - f(x)\, D_n(x)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin w/2} + f(x) D_n(x)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin\frac{w}{2}} - D_n(x)\, f(x)\ \frac{\sin x/2}{2\sin w/2} \Big]$$

$$= \lim_{w\to 0^-} \big(f(x+w) D_n(x+w) - f(x) D_n(x)\big)\ \frac{\sin\frac{1}{2}(w+x)}{2\sin w/2}\ \ \lim_{w\to 0^-} f(x) D_n(x)\ \frac{\sin\frac{w+x}{2} - \sin\frac{x}{2}}{2\sin w/2}$$

$$= \lim_{w\to 0^-} \underbrace{\frac{f(x+w) D_n(x+w) - f(x) D_n(x)}{w}\ \cdot \frac{w}{2\sin w/2}\ \ \sin\frac{1}{2}(w+x)\ \ f(x) D_n(x) \lim \frac{\sin\frac{w}{2}\cos(\frac{w}{2} \ldots)}{2\sin\frac{w}{2}}}_{①}$$

f پیوسته است پس ... چپ نقطه از ... تا پیوستگی از نوع پیوند ثابت ندارد ... یا $\lim_{w\to 0^-} \frac{f(x+w) - f(x)}{w}$ یا

... $\frac{f(x+ \ldots) - f(x)}{\ldots}$ ... وجود دارند ... در ① و ② ... وجود دارد در ... حد و از جپ متناهی است
$\to 0^+$

... ترتیب ... و ... پس g پیوسته ای است ... (مجذور ... سوال که ای هموار بودن ... ذکر شده.)

اثبات

۲) ... g ... حاصل ... تابع متناوب ... $f(x)$ ... $\frac{\sin\frac{w+x}{2}}{\sin\frac{w}{2}}$ است پس و متناوب

است (اما ... و ...)

تابع f در فضای در نقطه x_0 در شرط لیپ شیتز از مرتبه α صدق می‌کند هرگاه اعداد مثبت M و δ موجود باشند

بطوری که $|f(x) - f(x_0)| \leq M|x-x_0|^\alpha$ مشروط به اینکه $|x-x_0|<\delta$. ثابت کنید اگر f پیوسته بوده و در

شرط لیپ شیتز در x_0 صدق کند آنگاه سری فوریه f به سمت $f(x_0)$ همگراست.

برهان: $x - x_0 = t \Rightarrow x = t + x_0$ پیوستار

$$|t| < \delta \Rightarrow |f(x_0+t) - f(x_0)| \leq M|t|^{\alpha} \quad \text{if} \quad \alpha = 1 \Rightarrow |f(x+t) - f(x_0)| \leq M|t|$$

$$f \text{ مجموع جزئی ام سری فوریه } = S_n(x) = \sum_{k=-n}^{n} C_k e^{ikx}, \quad C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt$$

$$S_n(x) = \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \right) e^{ikx} = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{ik(x-t)}\, dt$$

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$$

* با توجه به تعریف ارائه شده در سوال ۲۴ داریم:

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{k=-n}^{n} e^{ik(x-t)}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(x-t)\, dt$$

$$\left.\begin{array}{l} u = x - t \Rightarrow t = x - u \\ du = -dt \end{array}\right\} \Rightarrow S_n(x) = \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-u) D_n(u)(-du) = \frac{1}{2\pi}\int_{-\pi+x}^{\pi+x} f(x-u) D_n(u)\, du$$

با توجه به تمرین ۲۳ ص ۵۹

$$\int_a^{a+2p} f(x)\, dx = \int_{-p}^{p} f(x)\, dx \Rightarrow S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_n(t)\, dt$$

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-t) - f(x) \right) D_n(t)\, dt$$

$$, D_n(t) = \sum_{k=-n}^{n} e^{ikt} = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}$$

با توجه به سوال ۲٤

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{2\sin t/2} \sin(n+\tfrac{1}{2})t\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{2\sin t/2} \left(\sin nt \cos t/2 + \cos nt \sin t/2 \right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{2\tan\frac{t}{2}} \sin nt\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-t) - f(x) \right) \cos nt\, dt = I_1 + I_2$$

الأسس ٣٥ -

$$|I_1| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{f(x-t) - f(x)}{2\tan t/2} \sin nt \right| dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \leq$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} M|\sin nt|\,dt \leq \frac{Mn}{\pi}\int_{0}^{\frac{\pi}{n}} \sin nt\,dt$$

$$|I_2| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\,|\cos nt|\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} M|t|\,|\cos nt|\,dt = \frac{M}{2\pi}\int_{-\pi}^{\pi} |t\cos nt|\,dt$$

$$= \frac{M}{2\pi}\left[\int_{-\pi}^{0} -t|\cos nt|\,dt + \int_{0}^{\pi} t|\cos nt|\,dt\right]$$

$$|I_2| \leq \frac{M}{2\pi}\left[\int_{0}^{\pi} t|\cos nt|\,dt - \int_{-\pi}^{0} t|\cos nt|\,dt\right] = \frac{M}{2\pi}\left[2n\int_{0}^{\frac{\pi}{2n}} t\cos nt\,dt - 2n\int_{-\frac{\pi}{2n}}^{0} t\cos nt\,dt\right]$$

$$= \frac{Mn}{\pi}\left[\left[\left(\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right)\right]_{0}^{\frac{\pi}{2n}} - \left[\left(\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right)\right]_{-\frac{\pi}{2n}}^{0}\right]$$

$$= \frac{Mn}{\pi}\left(\frac{\pi}{2n^2} + \frac{\pi}{2n^2}\right) = \frac{M}{n}$$

$$|I_1 + I_2| \leq |I_1| + |I_2| = \frac{M}{n} + \frac{M}{n}\int_{0}^{\frac{\pi}{n}} \frac{\sin nt}{n}\,dt = P(n) \quad \text{in } f \to \infty \Rightarrow P(n) \to 0$$

$$\Rightarrow \lim_{n\to\infty} |S_n(x) - f(x)| = 0 \qquad \Rightarrow \lim_{n\to\infty} S_n(x) = f(x)$$